Fueter polynomials in discrete Clifford analysis

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Abstract

Discrete Clifford analysis is a higher dimensional discrete function theory, based on skew Weyl relations. The basic notions are discrete monogenic functions, i.e. Clifford algebra valued functions in the kernel of a discrete Dirac operator. In this paper, we introduce the discrete Fueter polynomials, which form a basis of the space of discrete spherical monogenics, i.e. discrete monogenic, homogeneous polynomials. Their definition is based on a Cauchy-Kovalevskaya extension principle. We present the explicit construction for this discrete Fueter basis, in arbitrary dimension m and for arbitrary homogeneity degree k.

1 Introduction

Euclidean Clifford analysis is a function theory which provides a generalization to higher dimension of the theory of holomorphic functions in the complex plane, see e.g. [1, 3, 7, 8]. The theory focusses on monogenic functions, i.e. Clifford algebra valued null solutions of the Dirac operator $\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$, where $(e_1, \ldots, e_m)$ is an orthonormal basis of $\mathbb{R}^m$, underlying the construction of the Clifford algebra.

In the paper [4], a discrete counterpart of Euclidean Clifford analysis was constructed, based on the introduction of skew Weyl relations. The basic notions of a discrete Dirac operator, discrete monogenic functions and discrete spherical monogenics, i.e. discrete monogenic, homogeneous polynomials were given. Next, in [5], a Cauchy–Kovalevskaya extension theorem (short: CK extension) in discrete Clifford analysis was established. The basic idea behind the CK extension, see [9, 2], is to characterize solutions of suitable (systems of) PDE's by their restriction, sometimes together with the restrictions of some of their derivatives, to a submanifold of codimension one. A nice and detailed historical overview of the CK extension is given in [6].

In this paper, the CK extension procedure is used for the explicit construction of a basis of the space of discrete spherical monogenics. The paper is organized as follows: the next section contains a brief introduction to discrete Clifford analysis. The third section summarizes the theory of the discrete CK extension. In the fourth section the notions of a discrete CK product and discrete Fueter polynomials, which constitute a basis of the space of spherical monogenics, are introduced and the explicit construction of the discrete Fueter basis, in arbitrary dimension m and for arbitrary homogeneity degree k is presented and proven in detail.

2 Preliminaries of discrete Clifford Analysis

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with orthonormal basis $\mathbf{e}_j$, $j = 1, \ldots, m$. Define the equidistant lattice $\mathbb{Z}_h^m$ with general mesh width $h > 0$ over the space $\mathbb{R}^m$ by

$$\mathbb{Z}_h^m = \{(\ell_1 h, \ell_2 h, \ldots, \ell_m h) \mid (\ell_1, \ell_2, \ldots, \ell_m) \in \mathbb{Z}^m\}$$

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The co-ordinates of a Clifford vector $\underline{x}$ will thus be integer multiples of the mesh width h. In this discrete setting the partial derivatives ∂_{x_j} are replaced by the one-sided forward and backward differences $\Delta_j^{\pm}, j = 1, \ldots, m$, which act on a function f as

$$\Delta_j^+[f] = \frac{f(\cdot + h\mathbf{e}_j) - f(\cdot)}{h}, \qquad \Delta_j^-[f] = \frac{f(\cdot) - f(\cdot - h\mathbf{e}_j)}{h}$$

and are seen as 'lowering operators'. With respect to the $\mathbb{Z}_h^m$-neighbourhood of $\underline{x}$, the usual definition of the discrete Laplacian then explicitly reads

$$\Delta_h^*[f](\underline{x}) = \sum_{j=1}^{m} \Delta_j^+ \Delta_j^- [u] = \sum_{j=1}^{m} \left(\frac{f(\underline{x} + h\mathbf{e}_j) - f(\underline{x} - h\mathbf{e}_j)}{h^2} \right) - 2m \frac{f(\underline{x})}{h^2}$$

The notation Δ^* refers to this operator being called the "star Laplacian", since it contains function values at the midpoints of the faces of the m-cube centered at $\underline{x}$.

In order to take into account the orientation, included in the difference operators, also in the basis vectors, we embed the Clifford algebra $\mathbb{R}_{0,m}$ in a bigger complex Clifford algebra $\mathbb{C}_{2m}$, the underlying vector space of which has double the dimension. Each basis element $\mathbf{e}_j$ now can be split into a forward and backward basis element $\mathbf{e}_j^+$ and $\mathbf{e}_j^-$, satisfying the anti-commutator relations

$$\mathbf{e}_j^- \mathbf{e}_\ell^- + \mathbf{e}_\ell^- \mathbf{e}_j^- = 0, \ \mathbf{e}_j^+ \mathbf{e}_\ell^+ + \mathbf{e}_\ell^+ \mathbf{e}_j^+ = 0, \ \mathbf{e}_j^+ \mathbf{e}_\ell^- + \mathbf{e}_\ell^- \mathbf{e}_j^+ = \delta_{j\ell}$$

Note that these conditions imply that $\mathbf{e}_j^2 = +1$, $j = 1, \ldots, m$, in contrast to the usual continuous Clifford setting where traditionally $\mathbf{e}_j^2 = -1$ is chosen.

A central notion in discrete Clifford analysis are discrete monogenic functions, i.e. Clifford algebra valued functions in the kernel of the discrete Dirac operator D, which is defined by

$$D[f](\underline{x}) = \sum_{j=1}^{m} \left(\mathbf{e}_j^+ \Delta_j^+[f](\underline{x}) + \mathbf{e}_j^- \Delta_j^-[f](\underline{x}) \right)$$

The Dirac operator factorizes the star Laplacian $\left(D^2 = \Delta_h^*\right)$ and tends to $\underline{\partial}$ as $h \to 0$.

We then introduce 'raising operators' $X_j^{\pm}$, defined by their interaction with the corresponding 'lowering operators' $\Delta_j^{\pm}$, according to the skew Weyl relations

$$\Delta_j^+ X_j^+ - X_j^- \Delta_j^- = 1 \tag{1}$$

$$\Delta_j^- X_j^- - X_j^+ \Delta_j^+ = 1, \tag{2}$$

which replace the classical Weyl relations holding in the continuous case for the partial derivatives and the vector variable. Combining each $X_j^{\pm}$ with the corresponding forward/backward basis element $\mathbf{e}_j^{\pm}$, results in the discrete vector variable

$$X = \sum_{j=1}^{m} \left(\mathbf{e}_j^+ X_j^- + \mathbf{e}_j^- X_j^+ \right)$$

of which the components $X_j^{\pm}$ are no longer independent, but they are interconnected by (1)-(2).

The discrete Euler operator E has the explicit form

$$E = \sum_{j=1}^{m} \left(\mathbf{e}_j^+ \mathbf{e}_j^- X_j^- \Delta_j^- + \mathbf{e}_j^- \mathbf{e}_j^+ X_j^+ \Delta_j^+ \right)$$

and satisfies the same intertwining relations with the Dirac operator and the vector variable which also hold in the continuous Clifford setting, i.e.

$$DX + XD = 2E + m, \qquad DE = ED + D, \qquad EX = XE + X \tag{3}$$

The notion of homogeneity of a discrete polynomial is then defined as follows.

Definition 2.1 *A discrete polynomial P is called homogeneous of degree k if and only if it is an eigenfunction with eigenvalue k of the discrete Euler operator: $EP = kP$.*

By this definition and (3), the action of the operator X to a discrete homogeneous polynomial of degree k will result in a discrete homogeneous polynomial of degree $k+1$, as expected. Next defining a spherical discrete monogenic to be a discrete homogeneous polynomial null solution of D, we have derived in [4] that the dimension of the space $\mathcal{M}_k^{(m)}$ of spherical discrete monogenics of degree k on $\mathbb{Z}_h^m$ is given by

$$\dim(\mathcal{M}_k^{(m)}) = \frac{(k+m-2)!}{k!(m-2)!} \tag{4}$$

The introduction of co-ordinate variables $\xi_j = X_j^+ \mathbf{e}_j^- + X_j^- \mathbf{e}_j^+$ and co-ordinate difference operators $\partial_j = \mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^-$, $j = 1, \dots, m$, which respectively decompose the discrete vector variable and the discrete Dirac operator as $D = \sum_{j=1}^m \partial_j$ and $X = \sum_{j=1}^m \xi_j$, enables us to work simultaneously on the considered graph and on its dual. These co-ordinate variables and difference operators inherit their (anti)commutation relations from the skew Weyl relations (1)-(2), i.e. $\partial_j \xi_j - \xi_j \partial_j = 1$, $\partial_\ell \xi_j + \xi_j \partial_\ell = 0$, $\ell \neq j$.

Once again using the intertwining relation $EX = X(E+1)$, it directly follows that $E\xi_j = \xi_j(E+1)$, whence $\xi_j^k[1]$, i.e. natural powers of the operator ξ_j acting on the ground state 1, are the basic homogeneous discrete polynomials of degree k in the variable x_j, similar to the basic powers x_j^k in the continuous setting. The homogeneous discrete polynomials $\xi_j^k[1]$ are explicitly given by $\xi_j[1](x_j) = x_j\,(\mathbf{e}_j^+ + \mathbf{e}_j^-)$ and

$$\xi_j^{2n+1}[1](x_j) = x_j\,(\mathbf{e}_j^+ + \mathbf{e}_j^-) \prod_{i=1}^{n} (x_j^2 - i^2) \tag{5}$$

$$\xi_j^{2n}[1](x_j) = \left(x_j^2 + n x_j (\mathbf{e}_j^+ \mathbf{e}_j^- - \mathbf{e}_j^- \mathbf{e}_j^+)\right) \prod_{i=1}^{n-1} (x_j^2 - i^2) \tag{6}$$

In the following lemma, their fundamental properties are listed, see [4].

Lemma 2.1 *For all $k \in \mathbb{N}$ and $j, \ell = 1, \dots, m$ we have*

$$\begin{aligned} \partial_j\, \xi_j^k[1] &= k\, \xi_j^{k-1}[1] \\ \partial_\ell\, \xi_j^k[1] &= 0, \ \ell \neq j \\ \partial_j \xi_j^{k_1} \xi_\ell^{k_2}[1] &= k_1\, \xi_j^{k_1-1} \xi_\ell^{k_2}[1], \ \ell \neq j \end{aligned}$$

Moreover, for any two multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$ with $|\alpha| = |\beta|$

$$\partial_m^{\alpha_m} \dots \partial_1^{\alpha_1} \left(\xi_1^{\beta_1} \dots \xi_m^{\beta_m}\right)[1] = \begin{cases} \alpha! & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta \end{cases}$$

where we have put $\alpha! = \alpha_1!\,\alpha_2! \dots \alpha_m!$

3 The discrete Cauchy-Kovalevskaya extension

In the paper [5], the following problem was examined:
Let f be a discrete function in the variables $x_2, \dots, x_m$, defined on the grid $\mathbb{Z}_h^{m-1}$ and taking values in the algebra over $\{\mathbf{e}_2^+, \mathbf{e}_2^-, \dots, \mathbf{e}_m^+, \mathbf{e}_m^-\}$. Does there exist a (unique) discrete monogenic function F in the variables $x_1, \dots, x_m$, defined on the grid $\mathbb{Z}_h^m$ and taking values in the algebra over $\{\mathbf{e}_1^+, \mathbf{e}_1^-, \dots, \mathbf{e}_m^+, \mathbf{e}_m^-\}$, such that $F|_{x_1=0} = f$?

This problem can be considered as the Cauchy-Kovalevskaya extension (or CK extension) problem in discrete Clifford analysis. In [5], the answer was shown to be positive and an explicit construction was given, as described in the following definition.

Definition 3.1 *The CK extension of a discrete function $f(x_2, \ldots, x_m)$ is the discrete monogenic function*

$$CK[f](x_1, x_2, \ldots, x_m) = \sum_{k=0}^{\infty} \frac{\xi_1^k[1](x_1)}{k!} f_k(x_2, \ldots, x_m) \tag{7}$$

where $f_0 = f$ and $f_{k+1} = (-1)^{k+1} D' f_k$, where $D' = \sum_{j=2}^{m} \partial_j$ is the restriction of D to the hyperplane $x_1 = 0$.

Note that in this definition, no conditions are imposed on the original function f since from (5)-(6) it follows that

$$\begin{array}{rcll} \xi_1^{2n+1}[1](x_1) & = & 0 & \text{for } n \geqslant |x_1| \\ \xi_1^{2n}[1](x_1) & = & 0 & \text{for } n \geqslant |x_1| + 1, \end{array}$$

implying that for every point $(x_1, \ldots, x_m)$ of the grid $\mathbb{Z}^m$, there exists an $N \in \mathbb{N}$ such that all but the first N terms of the series in (7) vanish, whence the series reduces to a finite sum in every point of $\mathbb{Z}^m$. Thus, for any discrete function $f(x_2, \ldots, x_m)$, its CK extension is well-defined on $\mathbb{Z}_h^m$. Moreover it is unique, as was proven in [5].

For simplicity of notation, we will from now on denote $\xi_j^k[1]$ by ξ_j^k.

4 Discrete Fueter polynomials

The discrete CK extension procedure establishes a isomorphism between the space $\Pi_k^{(m-1)}$ of discrete homogeneous polynomials of degree k in $m-1$ variables and the space $\mathcal{M}_k^{(m)}$ of spherical discrete monogenics of degree k in m variables. Since the discrete homogeneous polynomials $\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}$, $\alpha_2 + \ldots + \alpha_m = k$ constitute a basis for the space $\Pi_k^{(m-1)}$, we have the foloowing result.

Theorem 4.1 *The set $\{CK[\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}] \mid \alpha_2 + \ldots + \alpha_m = k\}$ constitutes a basis for $\mathcal{M}_k^{(m)}$. The basis elements $CK[\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}]$ are called the discrete Fueter polynomials of degree k.*

The explicit construction of the discrete Fueter basis, in arbitrary dimension m and for arbitrary homogeneity degree k, is the subject of Section 4.2. However, we will first introduce, in the following section, the so–called Cauchy-Kovalevskaya product of discrete monogenic functions.

4.1 The Cauchy-Kovalevskaya product

Let f and g be discrete monogenic functions, defined on the whole of $\mathbb{Z}_h^m$, then their product fg will, in general, no longer be discrete monogenic. However, we may consider their restrictions to the hyperplane $x_1 = 0$, i.e. $f|_{x_1=0}$ and $g|_{x_1=0}$, which are discrete functions on $\mathbb{Z}_h^{m-1}$ with CK extensions f and g. Clearly, the function $f|_{x_1=0} \cdot g|_{x_1=0}$ also possesses a CK extension, which we denote by $f \odot g$ and call the discrete Cauchy-Kovalevskaya product (or: discrete CK product) of f and g, yielding a discrete monogenic function by definition. This discrete CK product then shows the following properties.

Lemma 4.1

(i) The discrete CK product is associative

(ii) If $f|_{x_1=0} \cdot g|_{x_1=0} = g|_{x_1=0} \cdot f|_{x_1=0}$ then $f \odot g = g \odot f$.

(iii) $1 \odot f = f \odot 1 = f$.

Proof Clearly, (ii) and (iii) directly follow from the definition of the discrete CK product, while for (i), application of the definition of the discrete CK product shows that

$$
\begin{aligned}
(f \odot g) \odot h &= \mathrm{CK}\left[f|_{x_1=0} \cdot g|_{x_1=0}\right] \odot h \;=\; \mathrm{CK}\left[\mathrm{CK}\left[f|_{x_1=0} \cdot g|_{x_1=0}\right]\Big|_{x_1=0} \cdot h|_{x_1=0}\right] \\
&= \mathrm{CK}\left[\left(f|_{x_1=0} \cdot g|_{x_1=0}\right) \cdot h|_{x_1=0}\right] \;=\; \mathrm{CK}\left[f|_{x_1=0} \cdot \left(g|_{x_1=0} \cdot h|_{x_1=0}\right)\right] \\
&= \mathrm{CK}\left[f|_{x_1=0} \cdot \mathrm{CK}\left[g|_{x_1=0} \cdot h|_{x_1=0}\right]\Big|_{x_1=0}\right] \\
&= \mathrm{CK}\left[f|_{x_1=0} \cdot (g \odot h)|_{x_1=0}\right] = f \odot (g \odot h)
\end{aligned}
$$

□

As can be seen from the following example, we will need in what follows an involution changing the sign of the co-ordinate variable ξ_1 and leaving the other co-ordinate variables unaltered. We thus define the involution $\hat{}$ by $\hat{\xi_1} = -\xi_1$, $\hat{\xi_j} = \xi_j$ for $j = 2, \ldots, m$ and the product rule $\widehat{ab} = \hat{a}\hat{b}$.

Example 4.1 *The CK extension of ξ_i $(i = 2, \ldots, m)$ is $z_i = \xi_i - \xi_1$, hence $\hat{z}_i = \xi_i + \xi_1$; the CK extension of $\xi_i\xi_j$ is $\frac{1}{2}(z_i z_j - z_j z_i)$ if $i \neq j$ or $z_i \hat{z}_i$ if $i = j$. This means that*

$$
\begin{aligned}
z_i \odot z_j &= \frac{1}{2!}(z_i z_j - z_j z_i), \;\; \textit{if } i \neq j \\
z_i \odot z_i &= z_i \hat{z}_i
\end{aligned}
$$

The CK product of z_i and z_j, for $i \neq j$, indeed is anticommutative since $\xi_i\xi_j = -\xi_j\xi_i$.

4.2 Construction of the discrete Fueter polynomials

In what follows we will need some additional notations. Let $2 \leqslant \ell_1, \ldots, \ell_k \leqslant m$, then

- $(z_{\ell_1} \ldots z_{\ell_k})^{\mathrm{O}_j}$ denotes that every first, third, fifth,… occurrence of z_j is replaced by $\hat{z}_j$; furthermore $\mathrm{O}_{r_1, r_2, \ldots}$ denotes the composition of $\mathrm{O}_{r_1}, \mathrm{O}_{r_2}, \ldots$;
- $(z_{\ell_1} \ldots z_{\ell_k})^{\mathrm{E}_j}$ denotes that every second, fourth, sixth,… occurrence of z_j is replaced by $\hat{z}_j$; furthermore $\mathrm{E}_{r_1, r_2, \ldots}$ denotes the composition of $\mathrm{E}_{r_1}, \mathrm{E}_{r_2}, \ldots$;
- $(z_{\ell_1} \ldots z_{\ell_k})^{*_j}$ denotes that every z_j is replaced by $\hat{z}_j$ (and every $\hat{z}_j$ is replaced by z_j); furthermore $*_{r_1, r_2, \ldots}$ denotes the composition of $*_{r_1}, *_{r_2}, \ldots$.

Definition 4.1 *For a k-tuple of integers $(\ell_1, \ldots, \ell_k)$, with $\ell_j \in \{2, \ldots, m\}$, we put*

$$
V_{\ell_1, \ldots, \ell_k} = \frac{1}{k!} \sum_{\pi(\ell_1, \ldots, \ell_k)} \mathrm{sgn}(\pi) \; \left(z_{\pi(\ell_1)} \ldots z_{\pi(\ell_k)}\right)^{\mathrm{E}_{2, \ldots, m}}
$$

where the sum runs over all distinguishable permutations π of $(\ell_1, \ldots, \ell_k)$, and where by $\mathrm{sgn}(\pi)$ we denote the sign of the permutation π, i.e. $\mathrm{sgn}(\pi) = +1$ if the permutation π of $(\ell_1, \ldots, \ell_k)$ is even and -1 if the permutation is odd.

Theorem 4.2 *Let $\underline{\alpha} = (\alpha_2, \ldots, \alpha_m) \in \mathbb{N}^{m-1}$ be an $(m-1)$-tuple of integers and identify $\underline{\alpha}$ with $(\ell_1, \ell_2, \ldots, \ell_k)$, where $k = \alpha_2 + \ldots + \alpha_m$, $\ell_j \in \{2, \ldots, m\}$, $\ell_i \leqslant \ell_j$ for $i < j$ and the number of times that j appears in $(\ell_1, \ell_2, \ldots, \ell_k)$ equals α_j, $j = 2, \ldots, m$. Then*

$$
z_2^{\alpha_2} \odot \ldots \odot z_m^{\alpha_m} \equiv z_{\ell_1} \odot \ldots \odot z_{\ell_k} = \alpha_2! \ldots \alpha_m! \, V_{\ell_1, \ldots, \ell_k}.
$$

We will defer the proof of this theorem in the most general case to Section 4.3. To fix the ideas however, we will first prove the corresponding result for the special cases where

- $\alpha_i = 1$, $\forall i = 2, \ldots, m$
- only one $\alpha_j \neq 0$.

Proposition 4.1 *For $\underline{\alpha} = (1, \ldots, 1)$, the CK extension of $\xi^{\underline{\alpha}}$ is given by*

$$z_2 \odot z_3 \odot \ldots \odot z_m \equiv CK[\xi_2 \xi_3 \ldots \xi_m] = \frac{1}{(m-1)!} \sum_{\sigma \in S_{\underline{\alpha}}} \mathrm{sgn}(\sigma)\, z_{\sigma(2)} \ldots z_{\sigma(m)}$$

where $S_{\underline{\alpha}}$ is the set of all distinguishable permutations of $(\ell_1, \ldots, \ell_{m-1}) = (2, 3, \ldots, m)$.

It is clear that this proposition is consistent with Theorem 4.2 above. Indeed, if $\underline{\alpha} = (1, \ldots, 1)$, then $(\ell_1, \ldots, \ell_k) = (2, 3, \ldots, m)$. There are $(m-1)!$ distinguishable permutations, whence

$$V_{2,\ldots,m} = \frac{1}{(m-1)!} \sum_{\sigma(2,\ldots,m)} \mathrm{sgn}(\sigma)\, z_{\sigma(2)} z_{\sigma(3)} \ldots z_{\sigma(m)}$$

and $\mathrm{CK}\,[\xi_2 \xi_3 \ldots \xi_m] = 1!\,1! \ldots 1!\, V_{2,\ldots,m} = V_{2,\ldots,m}$.

For the proof of this proposition, some auxiliary results are needed.

Lemma 4.2 *For every $k \in \mathbb{N}$ and $\ell_1, \ldots, \ell_k \neq 1$, it holds that*

$$\partial_1 \left(z_{\ell_1} \ldots z_{\ell_k} \right) = -z_{\ell_2} \ldots z_{\ell_k} - \hat{z}_{\ell_1} \partial_1 \left(z_{\ell_2} \ldots z_{\ell_k} \right)$$
$$\partial_1 \left(\hat{z}_{\ell_1} \ldots z_{\ell_k} \right) = z_{\ell_2} \ldots z_{\ell_k} - z_{\ell_1} \partial_1 \left(z_{\ell_2} \ldots z_{\ell_k} \right).$$

Lemma 4.3 *For $j, p = 2, \ldots, m$ with $j \neq p$ and for $\ell_1, \ldots, \ell_k \in \{2, \ldots, m\}$, it holds that*

$$\begin{aligned}
\partial_j \left(z_j z_{\ell_1} \ldots z_{\ell_k} \right) &= z_{\ell_1} \ldots z_{\ell_k} + \hat{z}_j \partial_j \left(z_{\ell_1} \ldots z_{\ell_k} \right) \\
\partial_j \left(\hat{z}_j z_{\ell_1} \ldots z_{\ell_k} \right) &= z_{\ell_1} \ldots z_{\ell_k} + z_j \partial_j \left(z_{\ell_1} \ldots z_{\ell_k} \right) \\
\partial_j \left(z_p z_{\ell_1} \ldots z_{\ell_k} \right) &= -z_p \partial_j \left(z_{\ell_1} \ldots z_{\ell_k} \right).
\end{aligned}$$

Now Proposition 4.1 can be proven.

Proof It can be easily seen that the restriction of

$$F = \frac{1}{(m-1)!} \sum_{\sigma \in S_{\underline{\alpha}}} \mathrm{sgn}(\sigma)\, z_{\sigma(2)} \ldots z_{\sigma(m)}$$

to the hyperplane $x_1 = 0$ equals $\xi_2 \xi_3 \ldots \xi_m$. It hence suffices to prove the monogenicity of F, i.e. to show that $DF = 0$. Therefore, we split the discrete Dirac operator into the co-ordinate difference operators. The operator ∂_1 acts on all the terms of F, resulting, on account of Lemma 4.2, into

$$\begin{aligned}
\partial_1 \left(z_{\sigma(2)} \ldots z_{\sigma(m)} \right) &= -z_{\sigma(3)} \ldots z_{\sigma(m)} + \hat{z}_{\sigma(2)} z_{\sigma(4)} \ldots z_{\sigma(m)} - \hat{z}_{\sigma(2)} \hat{z}_{\sigma(3)} z_{\sigma(5)} \ldots z_{\sigma(m)} \\
&\quad + \ldots + (-1)^{m-1} \hat{z}_{\sigma(2)} \hat{z}_{\sigma(3)} \ldots \hat{z}_{\sigma(m-1)}
\end{aligned} \tag{8}$$

Next, notice that for every $j = 2, \ldots, m$, z_j appears exactly once in each term of F whence ∂_j only acts on that particular factor in each term. Since $\left\{ \partial_{\sigma(2)}, \ldots, \partial_{\sigma(m)} \right\} = \{ \partial_2, \ldots, \partial_m \}$, $D \left(z_{\sigma(2)} z_{\sigma(3)} \ldots z_{\sigma(m)} \right)$ splits into

$$\begin{aligned}
\partial_{\sigma(2)} \left(z_{\sigma(2)} z_{\sigma(3)} \ldots z_{\sigma(m)} \right) &= z_{\sigma(3)} z_{\sigma(4)} \ldots z_{\sigma(m)} \\
\partial_{\sigma(3)} \left(z_{\sigma(2)} z_{\sigma(3)} \ldots z_{\sigma(m)} \right) &= -z_{\sigma(2)} z_{\sigma(4)} \ldots z_{\sigma(m)} \\
&\vdots \\
\partial_{\sigma(m)} \left(z_{\sigma(2)} z_{\sigma(3)} \ldots z_{\sigma(m)} \right) &= (-1)^m z_{\sigma(2)} z_{\sigma(3)} \ldots z_{\sigma(m-1)}
\end{aligned} \tag{9}$$

on account of Lemma 4.3. Combining (8)–(9), the action of D on F thus results in

$$\begin{aligned}
DF = &\sum_{\sigma \in S_{\underline{\alpha}}} \mathrm{sgn}(\sigma) \left(\hat{z}_{\sigma(2)} - z_{\sigma(2)} \right) z_{\sigma(4)} \ldots z_{\sigma(m)} - \sum_{\sigma \in S_{\underline{\alpha}}} \left(\hat{z}_{\sigma(2)} \hat{z}_{\sigma(3)} - z_{\sigma(2)} z_{\sigma(3)} \right) z_{\sigma(5)} \ldots z_{\sigma(m)} \\
&+ \ldots + (-1)^{m-1} \sum_{\sigma \in S_{\underline{\alpha}}} \left(\hat{z}_{\sigma(2)} \hat{z}_{\sigma(3)} \ldots \hat{z}_{\sigma(m-1)} - z_{\sigma(2)} z_{\sigma(3)} \ldots z_{\sigma(m-1)} \right)
\end{aligned} \tag{10}$$

We will now show that the right hand side of (10) equals zero, by considering separately the respective sums in the above expresssion.

First sum

The first sum can be rewritten as $2\sum_{\sigma\in S_{\underline{\alpha}}} \mathrm{sgn}(\sigma)\,\xi_1\, z_{\sigma(4)}\ldots z_{\sigma(m)}$. For every $\sigma\in S_{\underline{\alpha}}$, there is a unique $\pi\in S_{\underline{\alpha}}$ such that $\pi(2)=\sigma(3)$, $\pi(3)=\sigma(2)$, and $\pi(j)=\sigma(j)$ for $j=4,\ldots,m$. Then obviously $\mathrm{sgn}(\sigma)=-\mathrm{sgn}(\pi)$ and hence

$$\mathrm{sgn}(\sigma)\,\xi_1\, z_{\sigma(4)}\ldots z_{\sigma(m)}+\mathrm{sgn}(\pi)\,\xi_1\, z_{\pi(4)}\ldots z_{\pi(m)}=0$$

Pairing in this way all permutations, the total sum is seen to be zero.

Second sum

Using $\hat{z}_{\sigma(2)}=\xi_{\sigma(2)}+\xi_1$ and $z_{\sigma(2)}=\xi_{\sigma(2)}-\xi_1$, the second sum in (10) can be rewritten as

$$\sum_{\sigma\in S_{\underline{\alpha}}} 2\,\mathrm{sgn}(\sigma)\left[\xi_{\sigma(2)}\xi_1+\xi_1\xi_{\sigma(3)}\right] z_{\sigma(5)}\ldots z_{\sigma(m)}$$

For every $\sigma\in S_{\underline{\alpha}}$ there are unique permutations $\pi,\mu\in S_{\underline{\alpha}}$ such that

$$\begin{aligned}\pi\,(2,\ldots,m) &= (\sigma(3),\sigma(4),\sigma(2),\sigma(5),\ldots,\sigma(m))\\ \mu\,(2,\ldots,m) &= (\sigma(4),\sigma(2),\sigma(3),\sigma(5),\ldots,\sigma(m))\end{aligned}$$

Clearly $\mathrm{sgn}(\sigma)=\mathrm{sgn}(\pi)=\mathrm{sgn}(\mu)$, whence the sum of the corresponding three terms equals

$$2\,\mathrm{sgn}(\sigma)\left(\xi_{\sigma(2)}\xi_1+\xi_1\xi_{\sigma(3)}+\xi_{\sigma(3)}\xi_1+\xi_1\xi_{\sigma(4)}+\xi_{\sigma(4)}\xi_1+\xi_1\xi_{\sigma(2)}\right) z_{\mu(5)}\ldots z_{\mu(m)}$$

Since $\xi_i\xi_j=-\xi_j\xi_1$ for $i\neq j$, this expression vanishes, and so does the total sum.

Last sum

The terms of the last sum can be rewritten (up to a factor $(-1)^{m-1}$) as

$$\mathrm{sgn}(\sigma)\left[\xi_{\sigma(2)}\xi_{\sigma(3)}+\xi_{\sigma(2)}\xi_1+\xi_1\xi_{\sigma(3)}+\xi_1^2\right]\left(\hat{z}_{\sigma(4)}\ldots\hat{z}_{\sigma(m-1)}-z_{\sigma(4)}\ldots z_{\sigma(m-1)}\right)$$

First observe, by combining the permutations in pairs as above, that the terms of the form

$$\mathrm{sgn}(\sigma)\,\xi_1^2\left(\hat{z}_{\sigma(4)}\ldots\hat{z}_{\sigma(m-1)}-z_{\sigma(4)}\ldots z_{\sigma(m-1)}\right)$$

will cancel each other out when taking the sum over all $\sigma\in S_{\underline{\alpha}}$. Thus, there are three kinds of terms left:

- $\mathrm{sgn}(\sigma)\,\xi_{\sigma(2)}\xi_{\sigma(3)}\left(\hat{z}_{\sigma(4)}\ldots\hat{z}_{\sigma(m-1)}-z_{\sigma(4)}\ldots z_{\sigma(m-1)}\right)$
- $\mathrm{sgn}(\sigma)\,\xi_{\sigma(2)}\xi_1\left(\hat{z}_{\sigma(4)}\ldots\hat{z}_{\sigma(m-1)}+z_{\sigma(4)}\ldots z_{\sigma(m-1)}\right)$
- $\mathrm{sgn}(\sigma)\,\xi_1\xi_{\sigma(3)}\left(\hat{z}_{\sigma(4)}\ldots\hat{z}_{\sigma(m-1)}+z_{\sigma(4)}\ldots z_{\sigma(m-1)}\right)$

Also here it is easily seen that, taking the sum over all permutations σ, the considered terms will cancel each other out and the total sum will equal zero. Since the argumentation used for the last sum may directly be transferred to all other sums appearing in (10), it follows that $DF=0$. □

Proposition 4.2 *For $j=1,\ldots,m$ and $k\in\mathbb{N}$, it holds that*

$$\underbrace{z_j\odot z_j\odot\ldots\odot z_j}_{k}\equiv CK\left[\xi_j^k\right]=\underbrace{z_j\hat{z}_j z_j\ldots}_{k}$$

Proof Putting

$$F=\underbrace{z_j\hat{z}_j z_j\ldots}_{k}$$

it suffices to prove that F is discrete monogenic and that its restriction to $x_1=0$ equals ξ_j^k. The latter directly follows, since $F|_{x_1=0}=F|_{\xi_1=0}$. Next, we examine the action of

$D = \sum_{j=1}^{m} \partial_j$ on F; note that only ∂_1 and ∂_j will have a nontrivial action. Invoking Lemma 4.2–4.3, we directly see that the corresponding terms will cancel each other out:

$$\partial_1 \underbrace{(z_j \hat{z}_j z_j \ldots)}_{k} = -k \underbrace{\hat{z}_j z_j \ldots}_{k-1}, \quad \partial_j \underbrace{(z_j \hat{z}_j z_j) \ldots}_{k} = k \underbrace{\hat{z}_j z_j \ldots}_{k-1}$$

whence F indeed is discrete monogenic. □

As was the case for Proposition 4.1, also Proposition 4.2 clearly is consistent with the general formulation of Theorem 4.2. Indeed, if $\underline{\alpha} = (0, \ldots, 0, k, 0, \ldots, 0)$ with k on the jth position, then $(\ell_1, \ldots, \ell_k) = (j, j, \ldots, j)$. There is only one distinguishable permutation, namely the identity, implying that

$$V_{j,j,\ldots,j} = \frac{1}{k!} \underbrace{z_j \hat{z}_j z_j \ldots}_{k}$$

and $\mathrm{CK}\left[\xi_j^k\right] = k!\, V_{j,j,\ldots,j} = \underbrace{z_j \hat{z}_j z_j \ldots}_{k}$.

Theorem 4.2 may now be proven by a combination of the above two special cases. However, this approach leads to lengthy and quite involved calculations. Therefore we have followed a different and more elegant approach, which will be outlined in the following section. This alternative approach also includes, and thus confirms, the two special cases already treated.

4.3 An alternative approach for the general proof

The main aim of this section is to prove the following lemma, from which the structure of the discrete Fueter polynomials will then directly follow.

Lemma 4.4 *Let P_k be a discrete spherical monogenic of degree k. For every $1 \leqslant n \leqslant k$*

$$P_k = \frac{n!(k-n)!}{k!} \sum_{(\ell_1,\ldots,\ell_n)} V_{\ell_1,\ldots,\ell_n}\, \partial_{\ell_n} \ldots \partial_{\ell_1} P_k \tag{11}$$

where repetitions of the $\ell_j \in \{2, \ldots, m\}$ are allowed, but where every subset $\{\ell_1, \ldots, \ell_n\}$ only appears once in the sum.

Before proving this lemma, we first give two examples.

Example 4.2 *Take $m = 4$ and $P_3 = CK[\xi_2\xi_3\xi_4]$, then Proposition 4.1 learns that*

$$P_3 = \xi_2\xi_3\xi_4 - \xi_1\xi_3\xi_4 + \xi_1\xi_2\xi_4 - \xi_1\xi_2\xi_3$$

Now take for instance $n = 1$, then the right hand side in the above statement is given by

$$\frac{2!}{3!} \sum_{\ell_1} V_{\ell_1}\, \partial_{\ell_1} P_3 = \frac{1}{3} \left(V_2\, \partial_2 P_3 + V_3\, \partial_3 P_3 + V_4\, \partial_4 P_3\right)$$

which, by direct calculation, indeed is found to equal P_3.

Example 4.3 *As mentioned in the formulation of Lemma 4.4, in the sum*

$$\sum_{(\ell_1,\ell_2,\ell_3)} V_{\ell_1,\ell_2,\ell_3}\, \partial_{\ell_3}\partial_{\ell_2}\partial_{\ell_1} P_k$$

every subset $\{\ell_1, \ell_2, \ell_3\}$ of $\{2, \ldots, m\}^3$ may only be chosen once. For example, for $m = 3$ and for

$$P_3 = 2\, CK\left[\xi_2^2\xi_3\right] + CK\left[\xi_2^3\right] = 2\,\xi_2^2\xi_3 - 4\,\xi_1\xi_2\xi_3 - 5\,\xi_1\xi_2^2 - 2\,\xi_1^2\xi_3 + \xi_2^3 - 3\,\xi_1^2\xi_2 + \frac{5}{3}\,\xi_1^3$$

one can easily check that

$$P_3 = V_{2,2,2}\, \partial_2^3 P_k + V_{3,3,3}\, \partial_3^3 P_k + V_{2,2,3}\, \partial_3\partial_2^2 P_k + V_{2,3,3}\, \partial_3^2\partial_2 P_k = 6\, V_{2,2,2} + 4\, V_{2,2,3}$$

Proof

Step one

Let P_k be a discrete spherical monogenic, i.e. $DP_k = 0$ and $EP_k = kP_k$. The Euler operator E can be written as $\sum_{j=1}^{m} \xi_j \partial_j$, whence

$$\begin{aligned}
kP_k &= \xi_1 \partial_1 P_k + \sum_{i=2}^{m} \xi_i \partial_i P_k \\
&= \xi_1 \left(DP_k - \sum_{i=2}^{m} \partial_i P_k \right) + \sum_{i=2}^{m} \xi_i \partial_i P_k \\
&= \sum_{i=2}^{m} (\xi_i - \xi_1) \partial_i P_k \;=\; \sum_{i=2}^{m} z_i \partial_i P_k \;=\; \sum_{i=2}^{m} V_i \partial_i P_k
\end{aligned} \tag{12}$$

Step two

Unlike in the continuous setting, where $\partial_i P_k$ also is monogenic, enabling the use of an inductive argument, here $\partial_i P_k$ is no longer discrete monogenic:

$$D\left[\partial_i P_k\right] = \left(\sum_{j=1}^{m} \partial_j \right) \partial_i P_k = -\partial_i D P_k + 2\partial_i^2 P_k = 2\partial_i^2 P_k, \qquad i = 2, \ldots, m$$

However, invoking these results, we still can repeat the calculation of step 1 for $\partial_i P_k$, taking into account that it is homogeneous of degree $k-1$. We obtain

$$\begin{aligned}
(k-1)\, \partial_i P_k &= \xi_1 \partial_1 \partial_i P_k + \sum_{j=2}^{m} \xi_j \partial_j \partial_i P_k = \xi_1 \left(D(\partial_i P_k) - \sum_{j=2}^{m} \partial_j \partial_i P_k \right) + \sum_{j=2}^{m} \xi_j \partial_j (\partial_i P_k) \\
&= 2\xi_1 \partial_i^2 P_k + \sum_{j=2}^{m} z_j \partial_j \partial_i P_k = (\hat{z}_i - z_i)\, \partial_i^2 P_k + \sum_{j=2}^{m} z_j \partial_j \partial_i P_k \\
&= \hat{z}_i \partial_i^2 P_k + \sum_{j=2, j \neq i}^{m} z_j \partial_j \partial_i P_k
\end{aligned}$$

whence, using (12),

$$\begin{aligned}
P_k &= \frac{1}{k} \sum_{i=2}^{m} z_i \partial_i P_k = \frac{1}{k} \frac{1}{(k-1)} \sum_{i=2}^{m} z_i \left(\hat{z}_i \partial_i^2 P_k + \sum_{j=2, j\neq i}^{m} z_j \partial_j \partial_i P_k \right) \\
&= \frac{1}{k(k-1)} \left(\sum_{\ell_1=2}^{m} 2!\; V_{\ell_1,\ell_1} \partial_{\ell_1}^2 P_k + \sum_{(\ell_1,\ell_2), \ell_2 \neq \ell_1} (z_{\ell_1} z_{\ell_2} \partial_{\ell_2} \partial_{\ell_1} + z_{\ell_2} z_{\ell_1} \partial_{\ell_1} \partial_{\ell_2})\, P_k \right) \\
&= \frac{1}{k(k-1)} \left(\sum_{\ell_1=2}^{m} 2!\; V_{\ell_1,\ell_1} \partial_{\ell_1}^2 P_k + \sum_{(\ell_1,\ell_2), \ell_2 \neq \ell_1} 2!\; V_{\ell_1,\ell_2} \partial_{\ell_2} \partial_{\ell_1} P_k \right) \\
&= \frac{2!(k-2)!}{k!} \sum_{(\ell_1,\ell_2)} V_{\ell_1,\ell_2}\; \partial_{\ell_2} \partial_{\ell_1} P_k .
\end{aligned}$$

Induction step

Suppose that (11) holds for every $1 \leqslant n \leqslant p-1$, then we are left to prove it for $n = p$. To this end, we take a $(p-1)$-tuple $(\ell_{p-1}, \ldots, \ell_1)$ and denote the number of times that the natural number i appears in $(\ell_1, \ldots, \ell_{p-1})$ by α_i, $(i = 2, \ldots, m)$. Then $(\ell_{p-1}, \ldots, \ell_1)$ is the result of a certain permutation π acting on $(\underbrace{2, \ldots, 2}_{\alpha_2}, \underbrace{3, \ldots, 3}_{\alpha_3}, \ldots, \underbrace{m, \ldots, m}_{\alpha_m})$, i.e.

$$\partial_{\ell_{p-1}} \ldots \partial_{\ell_1} P_k = \mathrm{sgn}(\pi)\; \partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k$$

whence

$$\begin{aligned}\partial_{\ell_{p-1}} \ldots \partial_{\ell_1} P_k &= \operatorname{sgn}(\pi)\, \partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k \\ &= \frac{\operatorname{sgn}(\pi)}{k-p+1} \left(\xi_1\, D[\partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k] + \sum_{j=2}^{m} z_j\, \partial_j \partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k \right) \qquad (13)\end{aligned}$$

Now, for the action of the discrete Dirac operator D on $\partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k$, we must make a distinction between the numbers i for which α_i is even and those for which α_i is odd, since for each $r \in \mathbb{N}$

$$D\left[\partial_i^{2r} f\right] = \partial_i^{2r} D\,[f] \qquad (14)$$

$$D\left[\partial_i^{2r+1} f\right] = -\partial_i^{2r+1} D\,[f] + 2\partial_i^{2r+2} f \qquad (15)$$

We write the i's for which α_i is odd in increasing order and denote for each such i its corresponding place in that sequence by $s(i)$. For example, for $\underline{\alpha} = (2,2,3,3,4,5,5,5)$, there are two i's for which α_i is odd, namely $i = 4$ and $i = 5$, so we will have in that case that $s(4) = 1$ and $s(5) = 2$. Returning to the general case, it then follows from (14)–(15) and the monogeneity of P_k that

$$D\,[\partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k] = 2 \sum_{\alpha_j \text{ odd}} (-1)^{s(j)+1}\, \partial_2^{\alpha_2} \ldots \partial_j^{\alpha_j+1} \ldots \partial_m^{\alpha_m} P_k$$

whence we may rewrite the first term in (13) as

$$\xi_1\, D[\partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k] = \sum_{\alpha_j \text{ odd}} (-1)^{s(j)+1}\, (\hat{z}_j - z_j)\, \partial_2^{\alpha_2} \ldots \partial_j^{\alpha_j+1} \ldots \partial_m^{\alpha_m} P_k$$

So

$$\begin{aligned}&\partial_{\ell_{p-1}} \ldots \partial_{\ell_1} P_k \\ &\quad= \frac{\operatorname{sgn}(\pi)}{k-p+1} \left(\sum_{\alpha_j \text{ odd}} (-1)^{s(j)+1} \hat{z}_j\, \partial_2^{\alpha_2} \ldots \partial_j^{\alpha_j+1} \ldots \partial_m^{\alpha_m} P_k + \sum_{\alpha_j \text{ even}} z_j\, \partial_j \partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k \right) \\ &\quad= \frac{\operatorname{sgn}(\pi)}{k-p+1} \left(\sum_{\alpha_j \text{ odd}} \hat{z}_j\, \partial_j \partial_2^{\alpha_2} \ldots \partial_j^{\alpha_j} \ldots \partial_m^{\alpha_m} P_k + \sum_{\alpha_j \text{ even}} z_j\, \partial_j \partial_2^{\alpha_2} \ldots \partial_m^{\alpha_m} P_k \right) \\ &\quad= \frac{1}{k-p+1} \left(\sum_{\alpha_j \text{ odd}} \hat{z}_j\, \partial_j \partial_{\ell_{p-1}} \ldots \partial_{\ell_1} P_k + \sum_{\alpha_j \text{ even}} z_j\, \partial_j \partial_{\ell_{p-1}} \ldots \partial_{\ell_1} P_k \right) \qquad (16)\end{aligned}$$

Using (16) in combination with the induction hypothesis, it then follows that

$$\begin{aligned}P_k &= \frac{(p-1)!(k-p+1)!}{k!} \sum_{(\ell_1,\ldots,\ell_{p-1})} V_{\ell_1,\ldots,\ell_{p-1}}\, \partial_{\ell_{p-1}} \ldots \partial_{\ell_1} P_k \\ &= \frac{(k-p)!}{k!} \sum_{(\ell_1,\ldots,\ell_{p-1})} (p-1)!\, V_{\ell_1,\ldots,\ell_{p-1}} \left(\sum_{\alpha_j \text{ odd}} \hat{z}_j\, \partial_j \partial_{\ell_{p-1}} \ldots \partial_{\ell_1} + \sum_{\alpha_j \text{ even}} z_j\, \partial_j \partial_{\ell_{p-1}} \ldots \partial_{\ell_1} \right) P_k\end{aligned}$$

Combining the appropriate terms corresponding with $\partial_{\ell_p} \partial_{\ell_{p-1}} \ldots \partial_{\ell_1}$ then leads to

$$P_k = \frac{(k-p)!}{k!} \sum_{(\ell_1,\ldots,\ell_{p-1},\, \ell_p)} p!\, V_{\ell_1,\ldots,\ell_{p-1},\ell_p}\, \partial_{\ell_p} \partial_{\ell_{p-1}} \ldots \partial_{\ell_1} P_k$$

□

As a corollary of Lemma 4.4, we are now able to prove theorem (4.2).

Proof of Theorem (4.2)
Let $\underline{\alpha} = (\alpha_2, \ldots, \alpha_m)$ with $\alpha_2 + \ldots + \alpha_m = k$. Identifying, as before, $\underline{\alpha}$ with the k-tuple $(\ell_1, \ldots, \ell_k)$, if then suffices to show that $P_k \equiv \mathrm{CK}\,[\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}] = \alpha_2! \ldots \alpha_m!\; V_{\ell_1,\ldots,\ell_k}$. On account of Lemma 4.4 we already have that

$$P_k = \sum_{(n_1,\ldots,n_k)} V_{n_1,\ldots,n_k}\, \partial_{n_k} \ldots \partial_{n_1} P_k \tag{17}$$

On the other hand, we have by definition that

$$P_k = \sum_{r=0}^{k} \frac{\xi_1^r}{r!} f_r = f_0 + \xi_1 f_1 + \frac{\xi_1^2}{2!} f_2 + \frac{\xi_1^3}{3!} f_3 + \ldots + \frac{\xi_1^k}{k!} f_k$$

where $f_0 = \xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}$ and $f_{r+1} = (-1)^{r+1} D' f_r$. In the above expression, each term is homogeneous of degree k and moreover, each term except f_0 contains at least one factor ξ_1. Therefore, the action of any operator of the form $\partial_{n_k} \ldots \partial_{n_1}$ on such a term will vanish, since $\partial_{n_k} \ldots \partial_{n_1}$ does not contain ∂_1. The only term at the right hand side of (17) yielding a result different from zero thus is

$$\partial_{\ell_k} \ldots \partial_{\ell_1} P_k = \partial_{\ell_k} \ldots \partial_{\ell_1} [f_0] = \partial_{\ell_k} \ldots \partial_{\ell_1} [\xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}] = \alpha_2! \ldots \alpha_m!$$

meaning that indeed

$$P_k = \alpha_2! \ldots \alpha_m!\; V_{\ell_1,\ldots,\ell_k}$$

$\square$

Example 4.4 *Since $\underline{\alpha} = (2, 1)$ can be identified with $(\ell_1, \ell_2, \ell_3) = (2, 2, 3)$, we have that*

$$CK\left[\xi_2^2 \xi_3\right] = z_2 \odot z_2 \odot z_3 = 2!\, V_{2,2,3} = \frac{1}{3}\left(z_2 \hat{z}_2 z_3 - z_2 z_3 \hat{z}_2 + z_3 z_2 \hat{z}_2\right)$$

Note that Lemma 4.4 is the discrete counterpart of a result in the continuous Clifford analysis setting (see [1, 11.2.5]), concerning the expansion of functions in Taylor series. It thus may also be seen as the first step towards the expansion of discrete functions in discrete Taylor series, which is a topic of future research.

5 Commutators of CK and ∂_j^k

In the continuous Clifford case, it holds for any analytic function f and any multi-index $\beta = (\beta_2, \ldots, \beta_m) \in \mathbb{N}^{m-1}$ that

$$\partial_{x_2}^{\beta_2} \ldots \partial_{x_m}^{\beta_m}\, \mathrm{CK}\,[f] = \mathrm{CK}\left[\partial_{x_2}^{\beta_2} \ldots \partial_{x_m}^{\beta_m}\, f\right]$$

in the appropriate region, since the commutator of the CK extension and any partial derivative ∂_{x_j}, $j = 2, \ldots, m$, is proven to be zero (see [1]). In the following section, we will examine the commutator of CK and the operator ∂_j^k in our setting. Unlike the continuous case, this commutator will only be zero in half of the cases, i.e. for even k.

5.1 Commutator of CK and ∂_j^2

Theorem 5.1 *Let $\underline{\alpha} = (\alpha_2, \ldots, \alpha_m)$ and $\xi^{\underline{\alpha}} = \xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}$. For $j = 2, \ldots, m$, it holds that*

$$CK\left[\partial_j^2\, \xi^{\underline{\alpha}}\right] = \partial_j^2\, CK\,[\xi^{\underline{\alpha}}]$$

Proof For $j = 2, \ldots, m$

$$D\left[\partial_j^2 f\right] = \partial_j^2 D\,[f]$$

whence $\partial_j^2\, \mathrm{CK}\,[\xi^{\underline{\alpha}}]$ is monogenic. Furthermore,

$$\left(\partial_j^2\, \mathrm{CK}\,[\xi^{\underline{\alpha}}]\right)\Big|_{x_1=0} = \partial_j^2\, \left(\mathrm{CK}\,[\xi^{\underline{\alpha}}]\right)\big|_{x_1=0} = \partial_j^2\, \xi^{\underline{\alpha}}$$

Because of the uniqueness of the CK extension, this implies that $\mathrm{CK}\left[\partial_j^2\, \xi^{\underline{\alpha}}\right] = \partial_j^2\, \mathrm{CK}\,[\xi^{\underline{\alpha}}]$. $\square$

5.2 Commutator of CK and ∂_j

A simple example shows that CK and ∂_j do not always commute as operators:

$$\partial_2\,\mathrm{CK}\left[\xi_2^2\right] = \partial_2\,(z_2\hat{z}_2) = 2\,\hat{z}_2, \qquad \mathrm{CK}\left[\partial_2\,\xi_2^2\right] = \mathrm{CK}\,[2\,\xi_2] = 2\,z_2$$

However, this example may suggest that, in order to obtain the CK extension of $\partial_2\,\xi^{\underline{\alpha}}$, we have to determine $\partial_2\,\mathrm{CK}\,[\xi^{\underline{\alpha}}]$ and replace every z_2 by $\hat{z}_2$ and vica versa, an operation which we have denoted by ^{*_2}. The following preliminary result states that the commutator of ∂_p and ^{*_j} is zero (for $j, p \neq 1$), when acting on $\mathrm{CK}\,[\xi^{\underline{\alpha}}]$.

Lemma 5.1 *Let $\underline{\alpha} = (\alpha_2, \ldots, \alpha_m)$ and $\xi^{\underline{\alpha}} = \xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}$. For $j, p = 2, \ldots, m$ it then holds that*

$$\partial_p\,(CK\,[\xi^{\underline{\alpha}}])^{*_j} = (\partial_p\;CK\,[\xi^{\underline{\alpha}}])^{*_j}$$

Proof One can prove by induction on k that for all $\ell_1, \ldots, \ell_k \in \{2, \ldots, m\}$ and for all $j, p = 2, \ldots, m$

$$\partial_p\,((z_{\ell_1} \ldots z_{\ell_k})^{*_j}) = (\partial_p\,(z_{\ell_1} \ldots z_{\ell_k}))^{*_j}\;,$$

which also holds if a factor z_{ℓ_s} is replaced by $\hat{z}_{\ell_s}$. Since $\partial_j\,(z_{\ell_s}) = \partial_j\,(\hat{z}_{\ell_s})$, for $j \neq 1$, this does not affect the proof. $\square$

Theorem 5.2 *Let $\underline{\alpha} = (\alpha_2, \ldots, \alpha_m)$ and $\xi^{\underline{\alpha}} = \xi_2^{\alpha_2} \ldots \xi_m^{\alpha_m}$. For $j = 2, \ldots, m$ it then holds that*

$$CK\,[\partial_j\,\xi^{\underline{\alpha}}] = (\partial_j\;CK\,[\xi^{\underline{\alpha}}])^{*_j}$$

and vica versa

$$\partial_j\;CK\,[\xi^{\underline{\alpha}}] = (CK\,[\partial_j\,\xi^{\underline{\alpha}}])^{*_j}$$

Proof Invoking Lemma (5.1), it suffices to prove that $\mathrm{CK}\,[\partial_j\,\xi^{\underline{\alpha}}] = \partial_j\,(\mathrm{CK}\,[\xi^{\underline{\alpha}}])^{*_j}$. If $\alpha_j = 0$, then both sides of the previous equation obviously are zero. Furthermore, changing the order of the ξ_j's only changes the sign on both sides of the equality. We may thus, without loss of generality, assume that j equals the first index. Then, identifying as before $(\alpha_2, \ldots, \alpha_m)$ with $(\ell_1, \ldots, \ell_k)$, we have that $(\alpha_2 - 1, \ldots, \alpha_m)$ can be identified with $(\ell_2, \ldots, \ell_k)$. It now suffices to show that

$$\sum_{\sigma(\ell_2,\ldots,\ell_k)} \mathrm{sgn}(\sigma)\,\left(z_{\sigma(\ell_2)} \ldots z_{\sigma(\ell_k)}\right)^{E_{2,\ldots,m}} = \frac{1}{k} \sum_{\sigma(\ell_1,\ldots,\ell_k)} \mathrm{sgn}(\sigma)\;\partial_2\left(\left(z_{\sigma(\ell_1)} \ldots z_{\sigma(\ell_k)}\right)^{O_2,\,E_{3,\ldots,m}}\right)$$

Since ∂_2 only acts on the factors z_2 in each term at the RHS, we have that

$$\begin{aligned}&\partial_2\left(\left(z_{\sigma(\ell_1)} \ldots z_{\sigma(\ell_k)}\right)^{O_2,\,E_{3,\ldots,m}}\right)\\ &\quad= \sum_{\substack{r=1\\ \sigma(\ell_r)=2}}^{k} (-1)^{(r-1)-(\#z_2\text{'s before place } r)}\;\left(z_{\sigma(\ell_1)} \ldots z_{\sigma(\ell_{r-1})}\;z_{\sigma(\ell_{r+1})} \ldots z_{\sigma(\ell_k)}\right)^{E_{2,3,\ldots,m}}\end{aligned}$$

The action of ∂_2 on the $k - (\alpha_2 - 1)$ different terms

$$\sum_{\substack{s=2\\ \sigma(\ell_s)\neq 2}}^{k+1} \left(z_{\sigma(\ell_2)} \ldots z_{\sigma(\ell_{s-1})}\;z_2\;z_{\sigma(\ell_s)} \ldots z_{\sigma(\ell_k)}\right)^{O_2,\,E_{3,\ldots,m}}$$

thus results in k times the term $\left(z_{\sigma(\ell_2)} \ldots z_{\sigma(\ell_k)}\right)^{E_{2,\ldots,m}}$, which all have the same sign. Indeed, reconsider

$$\left(z_{\sigma(\ell_2)} \ldots z_{\sigma(\ell_{s-1})}\;z_2\;z_{\sigma(\ell_s)} \ldots z_{\sigma(\ell_k)}\right)^{O_2,\,E_{3,\ldots,m}}$$

with $\sigma(\ell_s) \neq 2$ and the factor z_2 on place $s - 1$. Let π be the permutation of $(\ell_1, \ldots, \ell_k)$ such that

$$\pi\,(\ell_1, \ldots, \ell_{s-2}, \ell_{s-1}, \ell_s, \ldots, \ell_k) = (\sigma(\ell_2), \ldots, \sigma(\ell_{s-1}), 2, \sigma(\ell_s), \ldots, \sigma\,(\ell_k))$$

Then $\left(z_{\sigma(\ell_2)} \ldots z_{\sigma(\ell_{s-1})}\ z_2\ z_{\sigma(\ell_s)} \ldots z_{\sigma(\ell_k)}\right)^{O_2,\, E_{3,\ldots,m}} = \left(z_{\pi(\ell_1)} \ldots z_{\pi(\ell_k)}\right)^{O_2,\, E_{3,\ldots,m}}$, and

$$\mathrm{sgn}(\pi) = (-1)^{(s-2)-(\#z_2\text{'s before place } s\,-\,1)}\ \mathrm{sgn}(\sigma)$$

After the action of ∂_2 on the factor on place $s-2$, we see that

$$\begin{aligned} &\mathrm{sgn}(\pi)\ (-1)^{(s-2)-(\#z_2\text{'s before place } s\,-\,1)}\ \left(z_{\pi(\ell_1)} \ldots z_{\pi(\ell_{s-2})}\ z_{\pi(\ell_s)} \ldots z_{\pi(\ell_k)}\right)^{E_{2,3,\ldots,m}} \\ &= \mathrm{sgn}(\sigma)\ \left(z_{\sigma(\ell_2)} \ldots z_{\sigma(\ell_{s-1})}\ z_{\sigma(\ell_s)} \ldots z_{\sigma(\ell_k)}\right)^{E_{2,3,\ldots,m}} \end{aligned}$$

the resulting sign thus only depending on σ and not on π. $\square$

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